Chapter 11

THREE DIMENSIONAL GEOMETRY

11.1 Overview

11.1.1 Direction cosines of a line are the cosines of the angles made by the line with positive directions of the co-ordinate axes.

11.1.2 If l, m, n are the direction cosines of a line, then $l^2 + m^2 + n^2 = 1$

11.1.3 Direction cosines of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

$$\frac{x_2 - x_1}{\text{PQ}}, \frac{y_2 - y_1}{\text{PQ}}, \frac{z_2 - z_1}{\text{PQ}},$$

where $\text{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

11.1.4 Direction ratios of a line are the numbers which are proportional to the direction cosines of the line.

11.1.5 If l, m, n are the direction cosines and a, b, c are the direction ratios of a line,

then $l \quad \frac{a}{\sqrt{a^2 \quad b^2 \quad c^2}}; m \quad \frac{b}{\sqrt{a^2 \quad b^2 \quad c^2}}; n \quad \frac{c}{\sqrt{a^2 \quad b^2 \quad c^2}}$

11.1.6 Skew lines are lines in the space which are neither parallel nor interesecting. They lie in the different planes.

11.1.7 Angle between skew lines is the angle between two intersecting lines drawn from any point (preferably through the origin) parallel to each of the skew lines.

11.1.8 If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two lines and θ is the acute angle between the two lines, then

$$\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

11.1.9 If a_1, b_1, c_1 and a_2, b_2, c_2 are the directions ratios of two lines and θ is the acute angle between the two lines, then

$$\cos\theta = \left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,.\sqrt{b_1^2 + b_2^2 + b_3^2}}\right|$$

11.1.10 Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

11.1.11 Equation of a line through a point (x_1, y_1, z_1) and having directions cosines l, m, n (or, direction ratios a, b and c) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \text{ or } \left(\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}\right).$$

11.1.12 The vector equation of a line that passes through two points whose positions vectors are $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

11.1.13 Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

11.1.14 If θ is the acute angle between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, then

θ is given by $\cos\theta = \dfrac{|\vec{b}_1 . \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$ or $\theta = \cos^{-1}\dfrac{|\vec{b}_1 . \vec{b}_2|}{|\vec{b}_1||\vec{b}_2|}$.

11.1.15 If $\dfrac{x - x_1}{l_1} = \dfrac{y - y_1}{m_1} = \dfrac{z - z_1}{n_1}$ and $\dfrac{x - x_2}{l_1} = \dfrac{y - y_2}{m_2} = \dfrac{z - z_2}{n_2}$ are equations of two lines, then the acute angle θ between the two lines is given by $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.

11.1.16 The shortest distance between two skew lines is the length of the line segment perpendicular to both the lines.

11.1.17 The shortest distance between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ is

$$\left|\frac{\vec{b}_1 \quad \vec{b}_2 \;.\; \vec{a}_2-\vec{a}_1}{\left|\vec{b}_1 \quad \vec{b}_2\right|}\right|.$$

11.1.18 Shortest distance between the lines: $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

11.1.19 Distance between parallel lines $\vec{r} \quad \vec{a}_1 \quad \vec{b}$ and $\vec{r}=\vec{a}_2+\lambda\vec{b}$ is

$$\left|\frac{\vec{b} \quad \vec{a}_2-\vec{a}_1}{\left|\vec{b}\right|}\right|.$$

11.1.20 The vector equation of a plane which is at a distance p from the origin, where $\hat{n}$ is the unit vector normal to the plane, is $\vec{r}.\hat{n}=p$.

11.1.21 Equation of a plane which is at a distance p from the origin with direction cosines of the normal to the plane as l, m, n is $lx + my + nz = p$.

11.1.22 The equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$ or $\vec{r}.\vec{n}=d$, where $d=\vec{a}.\vec{n}$.

11.1.23 Equation of a plane perpendicular to a given line with direction ratios a, b, c and passing through a given point (x_1, y_1, z_1) is $a\,(x-x_1)+b\,(y-y_1)+c\,(z-z_1)=0$.

11.1.24 Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0 .$$

11.1.25 Vector equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is $(\vec{r}-\vec{a}).\left[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})\right]=0$

11.1.26 Equation of a plane that cuts the co-ordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

11.1.27 Vector equation of any plane that passes through the intersection of planes $\vec{r}.\vec{n}_1=d_1$ and $\vec{r}.\vec{n}_2=d_2$ is $(\vec{r}.\vec{n}_1-d_1)+\lambda(\vec{r}.\vec{n}_2-d_2)=0$, where λ is any non-zero constant.

11.1.28 Cartesian equation of any plane that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is $(A_1x + B_1y + C_1z + D_1) + \lambda\ (A_2x + B_2y + C_2z + D_2) = 0$.

11.1.29 Two lines $\vec{r} \quad \vec{a}_1 \quad \vec{b}_1$ and $\vec{r} = \vec{a}_2+\lambda\vec{b}_2$ are coplanar if $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=0$

11.1.30 Two lines $\frac{x-x_1}{a_1}\ \frac{y-y_1}{b_1}\ \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}\ \frac{y-y_2}{b_2}\ \frac{z-z_2}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0 ,$$

11.1.31 In vector form, if θ is the acute angle between the two planes, $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$, then $\theta=\cos^{-1}\frac{|\vec{n}_1.\vec{n}_2|}{|\vec{n}_1|.|\vec{n}_2|}$

11.1.32 The acute angle θ between the line $\vec{r} \quad \vec{a} \quad \vec{b}$ and plane $\vec{r}.\vec{n} = d$ is given by

$$\sin\theta = \frac{|\vec{b}\,.\,\vec{n}|}{|\vec{b}|\,.\,|\vec{n}|}.$$

11.2 Solved Examples

Short Answer (S.A.)

Example 1 If the direction ratios of a line are 1, 1, 2, find the direction cosines of the line.

Solution The direction cosines are given by

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Here a, b, c are 1, 1, 2, respectively.

Therefore, $l = \dfrac{1}{\sqrt{1^2+1^2+2^2}},\ m = \dfrac{1}{\sqrt{1^2+1^2+2^2}},\ n = \dfrac{2}{\sqrt{1^2+1^2+2^2}}$

i.e., $l = \dfrac{1}{\sqrt{6}},\ m = \dfrac{1}{\sqrt{6}},\ n = \dfrac{2}{\sqrt{6}}$ i.e. $\pm\left(\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}\right)$ are D.C's of the line.

Example 2 Find the direction cosines of the line passing through the points P (2, 3, 5) and Q (–1, 2, 4).

Solution The direction cosines of a line passing through the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are

$$\frac{x_2-x_1}{\text{PQ}},\ \frac{y_2-y_1}{\text{PQ}},\ \frac{z_2-z_1}{\text{PQ}}.$$

Here $\text{PQ} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$= \sqrt{(-1-2)^2+(2-3)^2+(4-5)^2} = \sqrt{9+1+1} = \sqrt{11}$$

Hence D.C.'s are

$$\pm\left(\frac{-3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}\right) \text{ or } \pm\left(\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}\right).$$

Example 3 If a line makes an angle of 30°, 60°, 90° with the positive direction of *x*, *y*, *z*-axes, respectively, then find its direction cosines.

Solution The direction cosines of a line which makes an angle of α, β, γ with the axes, are $\cos\alpha$, $\cos\beta$, $\cos\gamma$

Therefore, D.C.'s of the line are cos30°, cos60°, cos90° i.e., $\pm\left(\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$

Example 4 The *x*-coordinate of a point on the line joining the points Q (2, 2, 1) and R (5, 1, –2) is 4. Find its *z*-coordinate.

Solution Let the point P divide QR in the ratio λ : 1, then the co-ordinate of P are

$$\left(\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$$

But *x*– coordinate of P is 4. Therefore,

$$\frac{5\lambda+2}{\lambda+1} = 4 \Rightarrow \lambda = 2$$

Hence, the *z*-coordinate of P is $\frac{-2\lambda+1}{\lambda+1} = -1$.

Example 5 Find the distance of the point whose position vector is $(2\hat{i}+\hat{j}-\hat{k})$ from the plane $\vec{r}\,.\,(\hat{i} - 2\hat{j} + 4\hat{k}) = 9$

Solution Here $\vec{a} = 2\hat{i}+\hat{j}-\hat{k}$, $\vec{n}\ \ \hat{i}-2\hat{j}\ \ 4\hat{k}$ and $d = 9$

So, the required distance is $\dfrac{\left|\left(2\hat{i}+\hat{j}-\hat{k}\right).\left(\hat{i}-2\hat{j}+4\hat{k}\right)-9\right|}{\sqrt{1+4+16}}$

$$= \frac{|2-2-4-9|}{\sqrt{21}} = \frac{13}{\sqrt{21}}.$$

Example 6 Find the distance of the point (– 2, 4, – 5) from the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Solution Here P (–2, 4, – 5) is the given point.

Any point Q on the line is given by $(3\lambda - 3, 5\lambda + 4, (6\lambda - 8)$,

$$\overrightarrow{PQ} = (3\lambda - 1)\,\hat{i} + 5\lambda\,\hat{j} + (6\lambda - 3)\,\hat{k}.$$

Since $\overrightarrow{PQ} \perp \left(3\hat{i} + 5\hat{j} + 6\hat{k}\right)$, we have

$$3\,(3\lambda - 1) + 5(\,5\lambda) + 6\,(6\lambda - 3\,) = 0$$

$$9\lambda + 25\lambda + 36\lambda = 21, \text{ i.e. } \lambda = \frac{3}{10}$$

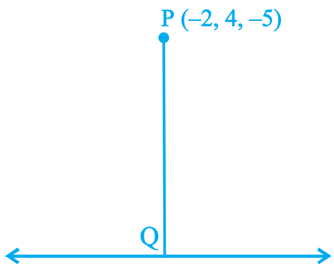

Fig. 11.1

Thus $\overrightarrow{PQ} = -\frac{1}{10}\hat{i} + \frac{15}{10}\hat{j} - \frac{12}{10}\hat{k}$

Hence $\left|\overrightarrow{PQ}\right| = \frac{1}{10}\sqrt{1+225+144} = \sqrt{\frac{37}{10}}.$

Example 7 Find the coordinates of the point where the line through (3, – 4, – 5) and (2, –3, 1) crosses the plane passing through three points (2, 2, 1), (3, 0, 1) and (4, –1, 0)

Solution Equation of plane through three points (2, 2, 1), (3, 0, 1) and (4, –1, 0) is

$$\left[(\vec{r} - (2\hat{i} + 2\hat{j} + \hat{k})\right].\left[(\hat{i} - 2\hat{j}) \times (\hat{i} - \hat{j} - \hat{k})\right] = 0$$

i.e. $\vec{r}.(2\hat{i} + \hat{j} + \hat{k}) = 7$ or $2x + y + z - 7 = 0$ ... (1)

Equation of line through (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} \qquad \text{... (2)}$$

Any point on line (2) is $(-\lambda + 3,\ \lambda - 4,\ 6\lambda - 5)$. This point lies on plane (1). Therefore, $2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) - 7 = 0$, i.e., $\lambda = z$

Hence the required point is $(1, -2, 7)$.

Long Answer (L.A.)

Example 8 Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}\,.\,(\hat{i} - \hat{j} + \hat{k}) = 5$.

Solution We have $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and $\vec{r}\,.\,(\hat{i} - \hat{j} + \hat{k}) = 5$

Solving these two equations, we get $[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - \hat{j} + \hat{k}) = 5$

which gives $\lambda = 0$.

Therefore, the point of intersection of line and the plane is $(2, \quad 1, 2)$ and the other given point is $(-1, -5, -10)$. Hence the distance between these two points is

$$\sqrt{2 \;\; (\;\; 1)^2 \quad [\;1 \;\; 5]^2 \quad [2 \;\; (\;\; 10)]^2}\text{, i.e. } 13$$

Example 9 A plane meets the co-ordinates axis in A, B, C such that the centroid of the Δ ABC is the point (α, β, γ). Show that the equation of the plane is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

Solution Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Then the co-ordinate of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Centroid of the Δ ABC is

$$\frac{x_1 \quad x_2 \quad x_3}{3}, \frac{y_1 \quad y_2 \quad y_3}{3}, \frac{z_1 \quad z_2 \quad z_3}{3} \quad \text{i.e.} \quad \frac{a}{3}, \frac{b}{3}, \frac{c}{3}$$

But co-ordinates of the centroid of the Δ ABC are (α, β, γ) (given).

Therefore, $\alpha = \frac{a}{3}, \beta = \frac{b}{3}, \gamma = \frac{c}{3}$, i.e. $a = 3\alpha$, $b = 3\beta$, $c = 3\gamma$

Thus, the equation of plane is

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$$

Example 10 Find the angle between the lines whose direction cosines are given by the equations: $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

Solution Eliminating m from the given two equations, we get

$\Rightarrow \quad 2n^2 + 3\,ln + l^2 = 0$

$\Rightarrow \quad (n + l)\,(2n + l) = 0$

$\Rightarrow \quad$ either $n = -\,l$ or $l = -\,2n$

Now if $\quad l = -\,n$, then $m = -\,2n$

and if $\quad l = -\,2n$, then $m = n$.

Thus the direction ratios of two lines are proportional to $-\,n, -2n, n$ and $-2n, n, n$,

i.e. $\quad 1, 2, -1$ and $-2, 1, 1$.

So, vectors parallel to these lines are

$$\vec{a} = \hat{i} + 2\hat{j} - \hat{k} \quad \text{and} \quad \vec{b} = -2\hat{i} + \hat{j} + \hat{k}, \text{ respectively.}$$

If θ is the angle between the lines, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{\left(\hat{i}+2\hat{j}-\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{1^2+2^2+(-1)^2}\,\sqrt{(-2)^2+1^2+1^2}} = -\frac{1}{6}$$

Hence $\quad \theta = \cos^{-1}\left(-\frac{1}{6}\right)$.

Example 11 Find the co-ordinates of the foot of perpendicular drawn from the point A (1, 8, 4) to the line joining the points B (0, –1, 3) and C (2, –3, –1).

Solution Let L be the foot of perpendicular drawn from the points A (1, 8, 4) to the line passing through B and C as shown in the Fig. 11.2. The equation of line BC by using formula $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, the equation of the line BC is

$$\vec{r} = \left(-\hat{j}+3\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}-4\hat{k}\right)$$

$$\Rightarrow \qquad x\hat{i} \quad y\hat{i} \quad z\hat{k} = 2 \quad \hat{i} - 2 \quad 1\ \hat{i} \qquad 3-4 \quad \hat{k}$$

Comparing both sides, we get

$$x = 2\lambda,\ y = -(2\lambda + 1),\ z = 3 - 4\lambda \qquad (1)$$

Thus, the co-ordinate of L are $(2\lambda, -(2\lambda + 1), (3 - 4\lambda)$,

so that the direction ratios of the line AL are $(1 - 2\lambda), 8 + (2\lambda + 1), 4 - (3 - 4\lambda)$, i.e.

$$1 - 2\lambda,\ 2\lambda + 9,\ 1 + 4\lambda$$

Since AL is perpendicular to BC, we have,

$$(1 - 2\lambda)(2 - 0) + (2\lambda + 9)(-3 + 1) + (4\lambda + 1)(-1 - 3) = 0$$

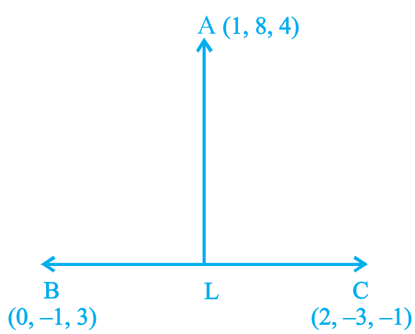

Fig.11.2

$$\Rightarrow \qquad \lambda = \frac{-5}{6}$$

The required point is obtained by substituting the value of λ, in (1), which is

$$\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right).$$

Example 12 Find the image of the point (1, 6, 3) in the line $\frac{x}{1} \quad \frac{y-1}{2} \quad \frac{z-2}{3}$.

Solution Let P (1, 6, 3) be the given point and let L be the foot of perpendicular from P to the given line.

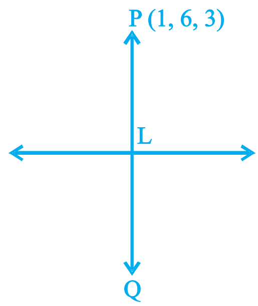

Fig.11.3

The coordinates of a general point on the given line are

$$\frac{x-0}{1} \quad \frac{y-1}{2} \quad \frac{z-2}{3} \qquad \text{, i.e., } x = \lambda,\ y = 2\lambda + 1,\ z = 3\lambda + 2.$$

If the coordinates of L are $(\lambda, 2\lambda + 1, 3\lambda + 2)$, then the direction ratios of PL are $\lambda - 1, 2\lambda - 5, 3\lambda - 1$.

But the direction ratios of given line which is perpendicular to PL are 1, 2, 3. Therefore, $(\lambda - 1)\,1 + (2\lambda - 5)\,2 + (3\lambda - 1)\,3 = 0$, which gives $\lambda = 1$. Hence coordinates of L are (1, 3, 5).

Let Q (x_1, y_1, z_1) be the image of P (1, 6, 3) in the given line. Then L is the mid-point of PQ. Therefore, $\frac{x_1 \quad 1}{2} \quad 1, \frac{y_1 \quad 6}{2} \quad 3 \quad \frac{z_1 \quad 3}{2} \quad 5$

$$\Rightarrow \qquad x_1 = 1,\ y_1 = 0,\ z_1 = 7$$

Hence, the image of (1, 6, 3) in the given line is (1, 0, 7).

Example 13 Find the image of the point having position vector $\hat{i} \quad 3\hat{j} \quad 4\hat{k}$ in the plane $\hat{r} \quad 2\hat{i}-\hat{j} \quad \hat{k} \quad 3 \quad 0$.

Solution Let the given point be P $\hat{i} \quad 3\hat{j} \quad 4\hat{k}$ and Q be the image of P in the plane $\hat{r} \,.\, 2\hat{i}-\hat{j} \quad \hat{k} \quad 3 \quad 0$ as shown in the Fig. 11.4.

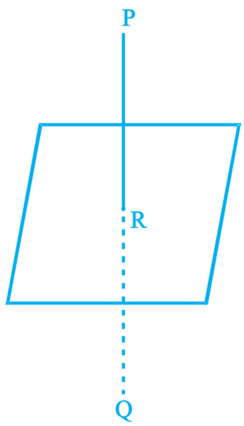

Fig.11.4

Then PQ is the normal to the plane. Since PQ passes through P and is normal to the given plane, so the equation of PQ is given by

$$\vec{r} = \hat{i} \quad 3\hat{j} \quad 4\hat{k} \qquad 2\hat{i}-\hat{j} \quad \hat{k}$$

Since Q lies on the line PQ, the position vector of Q can be expressed as

$$\hat{i} \quad 3\hat{j} \quad 4\hat{k} \qquad 2\hat{i}-\hat{j} \quad \hat{k} \text{ , i.e., } (1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}$$

Since R is the mid point of PQ, the position vector of R is

$$\frac{\left[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}\right]+\left[\hat{i}+3\hat{j}+4\hat{k}\right]}{2}$$

i.e., $$(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}$$

Again, since R lies on the plane $\vec{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)+3=0$, we have

$$\left\{(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}\right\}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$$

$\Rightarrow \qquad \lambda = -2$

Hence, the position vector of Q is $\left(\hat{i}+3\hat{j}+4\hat{k}\right)-2\ \ 2\hat{i}-\hat{j}\ \ \hat{k}$, i.e. $-3\hat{i}+5\hat{j}+2\hat{k}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 14 to 19.

Example 14 The coordinates of the foot of the perpendicular drawn from the point (2, 5, 7) on the x-axis are given by

(A) (2, 0, 0) (B) (0, 5, 0) (C) (0, 0, 7) (D) (0, 5, 7)

Solution (A) is the correct answer.

Example 15 P is a point on the line segment joining the points (3, 2, –1) and (6, 2, –2). If x co-ordinate of P is 5, then its y co-ordinate is

(A) 2 (B) 1 (C) –1 (D) –2

Solution (A) is the correct answer. Let P divides the line segment in the ratio of $\lambda : 1$, x - coordinate of the point P may be expressed as $x=\dfrac{6\lambda+3}{\lambda+1}$ giving $\dfrac{6\lambda+3}{\lambda+1}=5$ so that $\lambda = 2$. Thus y-coordinate of P is $\dfrac{2\lambda+2}{\lambda+1}=2$.

Example 16 If α, β, γ are the angles that a line makes with the positive direction of x, y, z axis, respectively, then the direction cosines of the line are.

(A) $\sin\alpha, \sin\beta, \sin\gamma$ (B) $\cos\alpha, \cos\beta, \cos\gamma$

(C) $\tan\alpha, \tan\beta, \tan\gamma$ (D) $\cos^2\alpha, \cos^2\beta, \cos^2\gamma$

Solution (B) is the correct answer.

Example 17 The distance of a point P (a, b, c) from x-axis is

(A) $\sqrt{a^2 \quad c^2}$ (B) $\sqrt{a^2 \quad b^2}$

(C) $\sqrt{b^2 \quad c^2}$ (D) $b^2 + c^2$

Solution (C) is the correct answer. The required distance is the distance of P (a, b, c) from Q (a, o, o), which is $\sqrt{b^2 \quad c^2}$.

Example 18 The equations of x-axis in space are

(A) $x = 0, y = 0$ (B) $x = 0, z = 0$ (C) $x = 0$ (D) $y = 0, z = 0$

Solution (D) is the correct answer. On x-axis the y- co-ordinate and z- co-ordinates are zero.

Example 19 A line makes equal angles with co-ordinate axis. Direction cosines of this line are

(A) $\pm (1, 1, 1)$ (B) $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

(C) $\pm\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ (D) $\pm\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

Solution (B) is the correct answer. Let the line makes angle α with each of the axis. Then, its direction cosines are $\cos \alpha, \cos \alpha, \cos \alpha$.

Since $\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$. Therefore, $\cos \alpha = \frac{1}{\sqrt{3}}$

Fill in the blanks in each of the Examples from 20 to 22.

Example 20 If a line makes angles $\frac{}{2}, \frac{3}{4}$ and $\frac{}{4}$ with x, y, z axis, respectively, then its direction cosines are ________

Solution The direction cosines are $\cos\frac{\ }{2}$, $\cos\frac{3}{4}$, $\cos\frac{\ }{4}$, i.e., $\pm\left(0, -\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\right)$.

Example 21 If a line makes angles α, β, γ with the positive directions of the coordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is _______

Solution Note that

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma)$$

$$= 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 2.$$

Example 22 If a line makes an angle of $\frac{\ }{4}$ with each of y and z axis, then the angle which it makes with x-axis is _________

Solution Let it makes angle α with x-axis. Then $\cos^2\alpha + \cos^2\frac{\ }{4} + \cos^2\frac{\ }{4} = 1$

which after simplification gives $\alpha = \frac{\ }{2}$.

State whether the following statements are **True** or **False** in Examples 23 and 24.

Example 23 The points (1, 2, 3), (–2, 3, 4) and (7, 0, 1) are collinear.

Solution Let A, B, C be the points (1, 2, 3), (–2, 3, 4) and (7, 0, 1), respectively.

Then, the direction ratios of each of the lines AB and BC are proportional to – 3, 1, 1.

Therefore, the statement is true.

Example 24 The vector equation of the line passing through the points (3,5,4) and (5,8,11) is

$$\overset{r}{r}\quad 3\hat{i}\quad 5\hat{j}\quad 4\hat{k}\qquad (2\hat{i}\quad 3\hat{j}\quad 7\hat{k})$$

Solution The position vector of the points (3,5,4) and (5,8,11) are

$$\overset{r}{a}\quad 3\hat{i}\quad 5\hat{j}\quad 4\hat{k}, \overset{r}{b}\quad 5\hat{i}\quad 8\hat{j}\quad 11\hat{k},$$

and therefore, the required equation of the line is given by

$$\overset{r}{r}\quad 3\hat{i}\quad 5\hat{j}\quad 4\hat{k}\qquad (2\hat{i}\quad 3\hat{j}\quad 7\hat{k})$$

Hence, the statement is true.

11.3 EXERCISE

Short Answer (S.A.)

1. Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60º to OX and at 45° to OY and $\left|\overrightarrow{OA}\right|$ = 10 units.

2. Find the vector equation of the line which is parallel to the vector $3\hat{i} \quad 2\hat{j} \quad 6\hat{k}$ and which passes through the point (1,–2,3).

3. Show that the lines

$$\frac{x \quad 1}{2} \quad \frac{y \quad 2}{3} \quad \frac{z \quad 3}{4}$$

and $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ intersect.

Also, find their point of intersection.

4. Find the angle between the lines

$$\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k}) \text{ and } \vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$$

5. Prove that the line through A (0, –1, –1) and B (4, 5, 1) intersects the line through C (3, 9, 4) and D (– 4, 4, 4).

6. Prove that the lines $x = py + q$, $z = ry + s$ and $x = p'y + q'$, $z = r'y + s'$ are perpendicular if $pp' + rr' + 1 = 0$.

7. Find the equation of a plane which bisects perpendicularly the line joining the points A (2, 3, 4) and B (4, 5, 8) at right angles.

8. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to coordinate axis.

9. If the line drawn from the point (–2, – 1, – 3) meets a plane at right angle at the point (1, – 3, 3), find the equation of the plane.

10. Find the equation of the plane through the points (2, 1, 0), (3, –2, –2) and (3, 1, 7).

11. Find the equations of the two lines through the origin which intersect the line

$\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

12. Find the angle between the lines whose direction cosines are given by the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

13. If a variable line in two adjacent positions has direction cosines l, m, n and $l + \delta l$, $m + \delta m$, $n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$$

14. O is the origin and A is (a, b, c).Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.

15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances a, b, c and a', b', c', respectively, from the origin, prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

Long Answer (L.A.)

16. Find the foot of perpendicular from the point (2,3,–8) to the line

$\frac{4\quad x}{2}\quad\frac{y}{6}\quad\frac{1\quad z}{3}$. Also, find the perpendicular distance from the given point to the line.

17. Find the distance of a point (2,4,–1) from the line

$$\frac{x\quad 5}{1}\quad\frac{y\quad 3}{4}\quad\frac{z\quad 6}{-9}$$

18. Find the length and the foot of perpendicular from the point $\left(1,\frac{3}{2},2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.

19. Find the equations of the line passing through the point (3,0,1) and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.

20. Find the equation of the plane through the points (2,1,–1) and (–1,3,4), and perpendicular to the plane $x - 2y + 4z = 10$.

21. Find the shortest distance between the lines given by $\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

22. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

23. The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. Prove that the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2+b^2} \tan\alpha) z = 0$.

24. Find the equation of the plane through the intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

25. Show that the points $(\hat{i} - \hat{j} + 3\hat{k})$ and $3(\hat{i} + \hat{j} + \hat{k})$ are equidistant from the plane $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$ and lies on opposite side of it.

26. $\overrightarrow{AB} = 3\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ are two vectors. The position vectors of the points A and C are $6\hat{i} + 7\hat{j} + 4\hat{k}$ and $-9\hat{j} + 2\hat{k}$, respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$ and $\overrightarrow{CD}$ both.

27. Show that the straight lines whose direction cosines are given by $2l + 2m - n = 0$ and $mn + nl + lm = 0$ are at right angles.

28. If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with them.

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises from 29 to 36.

29. Distance of the point (α,β,γ) from y-axis is

(A) β (B) $|\beta|$ (C) $|\beta|+|\gamma|$ (D) $\sqrt{\alpha^2+\gamma^2}$

30. If the directions cosines of a line are $k,k,k,$ then

(A) $k>0$ (B) $0<k<1$ (C) $k=1$ (D) $k \quad \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

31. The distance of the plane $\vec{r}\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ from the origin is

(A) 1 (B) 7 (C) $\frac{1}{7}$ (D) None of these

32. The sine of the angle between the straight line $\frac{x\quad 2}{3}\ \frac{y\quad 3}{4}\ \frac{z\quad 4}{5}$ and the plane $2x - 2y + z = 5$ is

(A) $\frac{10}{6\sqrt{5}}$ (B) $\frac{4}{5\sqrt{2}}$ (C) $\frac{2\sqrt{3}}{5}$ (D) $\frac{\sqrt{2}}{10}$

33. The reflection of the point (α,β,γ) in the xy– plane is

(A) $(\alpha,\beta,0)$ (B) $(0,0,\gamma)$ (C) $(-\alpha,-\beta,\gamma)$ (D) $(\alpha,\beta,-\gamma)$

34. The area of the quadrilateral ABCD, where A(0,4,1), B (2, 3, –1), C(4, 5, 0) and D (2, 6, 2), is equal to

(A) 9 sq. units (B) 18 sq. units (C) 27 sq. units (D) 81 sq. units

35. The locus represented by $xy + yz = 0$ is

(A) A pair of perpendicular lines (B) A pair of parallel lines

(C) A pair of parallel planes (D) A pair of perpendicular planes

36. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{\sqrt{2}}{3}$ (C) $\frac{2}{7}$ (D) $\frac{3}{7}$

Fill in the blanks in each of the Exercises 37 to 41.

37. A plane passes through the points (2,0,0) (0,3,0) and (0,0,4). The equation of plane is __________.

38. The direction cosines of the vector $(2\hat{i} \quad 2\hat{j} - \hat{k})$ are __________.

39. The vector equation of the line $\dfrac{x-5}{3} \quad \dfrac{y \quad 4}{7} \quad \dfrac{z-6}{2}$ is __________.

40. The vector equation of the line through the points (3,4,–7) and (1,–1,6) is __________.

41. The cartesian equation of the plane $\overset{r}{r}.(\hat{i} \quad \hat{j} - \hat{k}) \quad 2$ is __________.

State **True** or **False** for the statements in each of the Exercises 42 to 49.

42. The unit vector normal to the plane $x + 2y + 3z - 6 = 0$ is $\dfrac{1}{\sqrt{14}}\hat{i} \quad \dfrac{2}{\sqrt{14}}\hat{j} \quad \dfrac{3}{\sqrt{14}}\hat{k}$.

43. The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the co-ordinate axis are $-2, \dfrac{4}{3}, -\dfrac{4}{5}$.

44. The angle between the line $\vec{r} \quad (5\hat{i} - \hat{j} - 4\hat{k}) \quad (2\hat{i} - \hat{j} \quad \hat{k})$ and the plane $\vec{r}.(3\hat{i} - 4\hat{j} - \hat{k}) \quad 5 \quad 0$ is $\sin^{-1} \dfrac{5}{2\sqrt{91}}$.

45. The angle between the planes $\vec{r}.(2\hat{i} - 3\hat{j} \quad \hat{k}) \quad 1$ and $\bar{r}.(\hat{i} - \hat{j}) \quad 4$ is $\cos^{-1} \dfrac{-5}{\sqrt{58}}$.

46. The line $\vec{r} \quad 2\hat{i} - 3\hat{j} - \hat{k} \quad (\hat{i} - \hat{j} \quad 2\hat{k})$ lies in the plane $\vec{r}.(3\hat{i} \quad \hat{j} - \hat{k}) \quad 2 \quad 0$.

47. The vector equation of the line $\dfrac{x-5}{3} \quad \dfrac{y \quad 4}{7} \quad \dfrac{z-6}{2}$ is

$$\vec{r} \quad 5\hat{i} - 4\hat{j} \quad 6\hat{k} \qquad (3\hat{i} \quad 7\hat{j} \quad 2\hat{k}).$$

48. The equation of a line, which is parallel to $2\hat{i} \quad \hat{j} \quad 3\hat{k}$ and which passes through the point (5,–2,4), is $\dfrac{x-5}{2} \quad \dfrac{y \quad 2}{-1} \quad \dfrac{z-4}{3}$.

49. If the foot of perpendicular drawn from the origin to a plane is (5, – 3, – 2), then the equation of plane is $\vec{r}.(5\hat{i}-3\hat{j}-2\hat{k}) = 38$.